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Symmetry Lie algebra of the Dirac oscillator

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Abstract. Recently the name Dirac oscillator was given to a Dirac equation in which both the momenta and the coordinates appear linearly. In the non-relativistic limit, the equation satisfied by the large components is that of a standard harmonic oscillator plus a very strong spin-orbit coupling term. This equation can be solved exactly, and the spectrum presents degeneracies which on occasions are finite and on others infinite. We show that the symmetry Lie algebra is so(4)⊕so(3, 1) and find its generators explicitly.

1. Introduction

Some twenty years ago, Itô *et al* (1967) introduced a Dirac equation which, besides the momenta, is also linear in the coordinates, and can be solved exactly. Cook (1971) found its spectrum and showed that it presents rather unusual accidental degeneracies, which on occasions are finite and on others are infinite. Recently, the same equation was independently rederived by Moshinsky and Szczepaniak (1989), who called it the Dirac oscillator since, in the non-relativistic limit, the equation satisfied by the large components is that of a standard harmonic oscillator plus a very strong spin-orbit coupling term.

The latter equation was also considered by other authors (Ui and Takeda 1984, Balantekin 1985), who obtained it from different considerations and discussed the spectrum degeneracies from a supersymmetric viewpoint. Such an approach, however, does not lead to a full understanding of the degeneracies.

The purpose of the present paper is to show that the degeneracies of the Dirac oscillator can be explained by a standard symmetry Lie algebra although its generators are far from trivial. To construct them, we shall take advantage of the procedures recently implemented in the analysis of some simple two-dimensional problems (Moshinsky *et al* 1990). More specifically, we shall prove that the symmetry Lie algebra of the Dirac oscillator is so(4)⊕so(3, 1), where so(4) accounts for the finitely-degenerate levels, while so(3, 1) explains the infinitely-degenerate ones.

This paper is organised as follows. In section 2 the spectrum and accidental degeneracies of the Dirac oscillator Hamiltonian are reviewed. In section 3 the conditions to be imposed on the symmetry Lie algebra are listed and the nature of the latter suggested by an analysis of the degeneracies. In section 4, ladder operators

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connecting the degenerate eigenstates are obtained. In section 5, they are used to construct the generators of the symmetry Lie algebra and the latter is shown to fulfil all the requirements imposed in section 3. Finally, section 6 contains the conclusion.

2. Spectrum and degeneracies of the Dirac oscillator

The Dirac oscillator equation is defined by (Moshinsky and Szczepaniak 1989)

$$\mathrm{i}(\partial\psi/\partial t) = [\boldsymbol{\alpha}\cdot(\boldsymbol{p} - \mathrm{i}\boldsymbol{r}\beta) + m\beta]\psi \tag{2.1}$$

in units wherein $\hbar = c = m\omega = 1$. Here m is the mass of the particle, ω the frequency of the oscillator,

$$\boldsymbol{p} = (\hbar/\mathrm{i})\boldsymbol{\nabla} \qquad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.2}$$

and $\boldsymbol{\sigma}$ is the vector of Pauli spin matrices (Schiff 1955).

Let us now express the Dirac wavefunction ψ as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \exp(-\mathrm{i}Et/\hbar) \tag{2.3}$$

where ψ_1 and ψ_2 are respectively its time-independent large and small components. In the non-relativistic limit, the former satisfies the equation

$$H\psi_1 = \varepsilon\psi_1 \tag{2.4}$$

where the Hamiltonian is given by

$$H = \hat{N} - 2\boldsymbol{L}\cdot\boldsymbol{S} \tag{2.5}$$

in terms of the number of quanta and the orbital and spin angular momentum operators

$$\hat{N} = \tfrac{1}{2}(p^2 + r^2 - 3) \qquad \boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p} \qquad \boldsymbol{S} = \tfrac{1}{2}\boldsymbol{\sigma}. \tag{2.6}$$

The Hamiltonian (2.5) represents a standard harmonic oscillator plus a very strong spin-orbit coupling term, as its contribution to the total energy is of the same order as the separation between the oscillator levels.

Since H commutes with the total angular momentum operator

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} \tag{2.7}$$

its eigenfunctions can be expressed in spherical coordinates as

$$\psi_1 = \langle \boldsymbol{r}s | N(l\tfrac{1}{2})jm\rangle = \sum_{\mu\sigma} \langle l\mu, \tfrac{1}{2}\sigma | jm\rangle R_{Nl}(r)\, Y_{l\mu}(\theta, \varphi)\chi_\sigma(s) \tag{2.8}$$

where N denotes the eigenvalue of $\hat{N}$ and runs over $0, 1, 2, \ldots$, l and j are the orbital and total angular momentum quantum numbers respectively, $R_{Nl}(r)$ is the harmonic oscillator radial function (Moshinsky 1969), $Y_{l\mu}(\theta, \varphi)$ is a spherical harmonic, and $\chi_\sigma(s)$ a spinor (Rose 1957).

The energy spectrum is given by

$$\varepsilon_{Nlj} = N - [j(j+1) - l(l+1) - \tfrac{3}{4}] \tag{2.9}$$

and is plotted in figure 1. It exhibits degeneracies higher than the degeneracy $2j+1$ coming from the rotational invariance of the Hamiltonian. To study these so-called

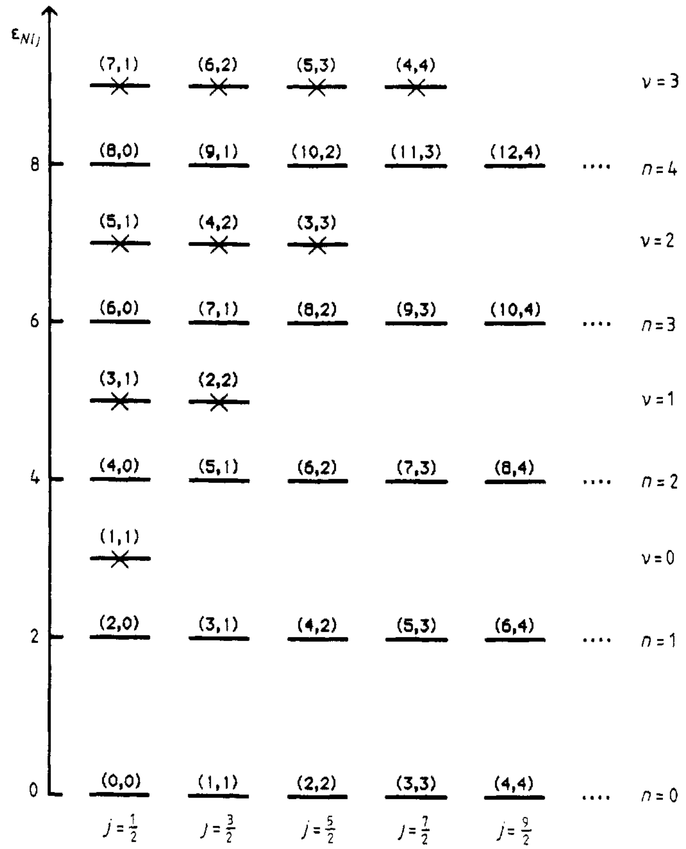

Figure 1. Energy spectrum of the Dirac oscillator. The abscissa corresponds to the total angular momentum and the ordinate to ε_{Nlj}. The levels are labelled by (N, l). Those belonging to $\mathscr{H}^{(+)}$ are marked by a cross, while those belonging to $\mathscr{H}^{(-)}$ are unmarked. The corresponding values of ν or n are indicated in the right column.

'accidental' degeneracies, it is convenient to divide the Hilbert space $\mathscr{H}$, spanned by the eigenfunctions (2.8), into two subspaces $\mathscr{H}^{(+)}$ and $\mathscr{H}^{(-)}$, containing the eigenfunctions with $l=j+\frac{1}{2}$ and $l=j-\frac{1}{2}$ respectively.

In $\mathscr{H}^{(+)}$, the spectrum is given by

$$\varepsilon_\nu = 2\nu + 3 \tag{2.10}$$

where

$$\nu = \tfrac{1}{2}(N + j - \tfrac{3}{2}) \tag{2.11}$$

runs over all non-negative integers. For any fixed value of ν, the corresponding level is made of sublevels with $j=\frac{1}{2}, \frac{3}{2}, \ldots, \nu+\frac{1}{2}$. Its degeneracy is therefore finite and equal to

$$d(\nu) = \sum_{j=1/2}^{\nu+1/2} (2j+1) = (\nu+1)(\nu+2). \tag{2.12}$$

We shall denote the degenerate eigenstates by

$$|\nu jm) \equiv |2\nu - j + \tfrac{3}{2}(j+\tfrac{1}{2}, \tfrac{1}{2})jm\rangle \tag{2.13}$$

where on the left-hand side we use a round bracket.

In $\mathscr{H}^{(-)}$, the spectrum is given by

$$\varepsilon_n = 2n \tag{2.14}$$

where the radial quantum number

$$n = \tfrac{1}{2}(N - j + \tfrac{1}{2}) \tag{2.15}$$

runs as usual over all non-negative integers. Any level is now made of sublevels with $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, and its degeneracy is therefore infinite. We shall denote the degenerate eigenstates by

$$|njm] \equiv |2n + j - \tfrac{1}{2}(j - \tfrac{1}{2}, \tfrac{1}{2})jm\rangle \tag{2.16}$$

where on the left-hand side we use a square bracket.

In the following sections we shall prove that the accidental degeneracies of the Dirac oscillator can be explained by the existence of a symmetry Lie algebra.

3. Symmetry Lie algebra

The conditions to be imposed on the symmetry Lie algebra for a Hamiltonian with accidental degeneracy were recently reviewed by Moshinsky *et al* (1990). In principle, the procedure for finding the symmetry Lie algebra goes as follows.

(i) Find ladder operators connecting all the eigenstates with a given energy.

(ii) Consider in addition the operators defined by the commutators of the ladder operators. If, together with the latter, they form a Lie algebra, then we have obtained a good candidate for the symmetry Lie algebra of the problem. If they do not, we may try to renormalise the ladder operators in such a way that closure is achieved, as is done, for instance, in the case of the hydrogen atom Hamiltonian (Fock 1935, Bargmann 1936).

(iii) Once a Lie algebra has been constructed, we expect that the set of eigenstates with given energy provides a basis for a definite, energy-dependent irreducible representation (irrep) of this Lie algebra. In particular, the dimension of the irreps should agree with the level degeneracies.

(iv) The Hamiltonian of the problem should be related to the Casimir operator(s) of the Lie algebra.

In practice, however, even in very simple problems of accidental degeneracy, a great variety of approaches need to be followed in the search for a symmetry Lie algebra (Moshinsky *et al* 1990). In particular, conditions (iii) and (iv) above do not necessarily follow from the fulfilment of conditions (i) and (ii). The construction of a Lie algebra satisfying all four requirements may sometimes demand that rather complicated combinations of renormalised ladder operators be considered. In such a case, it often proves convenient to start by guessing the nature of the Lie algebra from the dimension of its irreps, as given by the level degeneracies; thence condition (iii) is automatically satisfied.

In the present problem, the explanation of the finite degeneracies in $\mathscr{H}^{(+)}$ and of the infinite ones in $\mathscr{H}^{(-)}$ requires both a compact and a non-compact algebra. It is indeed well known (Wybourne 1974) that all the unitary irreps of a compact (non-compact) Lie algebra are finite (infinite) dimensional (apart from the trivial one-dimensional irrep). If both algebras are not to interfere, their generators must act exclusively in $\mathscr{H}^{(+)}$ or in $\mathscr{H}^{(-)}$, respectively. Hence this suggests looking for the direct sum $\mathscr{L}^{(+)} \oplus \mathscr{L}^{(-)}$ of a compact and a non-compact algebra. This type of separation reminds us what happens for the hydrogen atom Hamiltonian, whose symmetry Lie algebra is so(4)$\oplus$e(3)$\oplus$so(3, 1). In this case, the orthogonal so(4), the euclidian e(3), and the pseudo-orthogonal so(3, 1) algebras respectively act in the spaces of negative-, zero-, or positive-energy states.

To ensure that the generators of $\mathscr{L}^{(+)}$ ($\mathscr{L}^{(-)}$) do not act in $\mathscr{H}^{(-)}$ ($\mathscr{H}^{(+)}$), we must use restricted operators, or, in other words, operators of the form $P^{(+)}OP^{(+)}$ ($P^{(-)}OP^{(-)}$), where $P^{(+)}$ ($P^{(-)}$) is the projection operator on $\mathscr{H}^{(+)}$ ($\mathscr{H}^{(-)}$). Such projection operators can be easily constructed in terms of the operators

$$\hat{L} = [\boldsymbol{L}^2 + \tfrac{1}{4}]^{1/2} - \tfrac{1}{2} \qquad \hat{J} = [\boldsymbol{J}^2 + \tfrac{1}{4}]^{1/2} - \tfrac{1}{2} \tag{3.1}$$

whose eigenvalues are l and j respectively, and they are given by

$$P^{(+)} = \hat{L} - \hat{J} + \tfrac{1}{2} \qquad P^{(-)} = \hat{J} - \hat{L} + \tfrac{1}{2}. \tag{3.2}$$

Let us now turn our attention to the determination of the symmetry Lie subalgebras $\mathscr{L}^{(\pm)}$. In $\mathscr{H}^{(+)}$, the level degeneracies $d(\nu)$, given in (2.12), coincide with the dimensions of the so(4) (spin) irreps characterised by the Young pattern labels $[pq]$, where $p = \nu + \frac{1}{2}$, and $q = \frac{1}{2}$ (Biedenharn 1961). Moreover, since the so(3) (or more exactly su(2)) content of $[pq]$ is given by $|q|, |q|+1, \ldots, p$, it corresponds to the total angular momenta of the degenerate sublevels. Hence, this suggests identifying $\mathscr{L}^{(+)}$ with so(4), the eigenstates (2.13) belonging to $\mathscr{H}^{(+)}$ with the basis states of $[\nu + \frac{1}{2}, \frac{1}{2}]$, and the generators of the so(3)$\simeq$su(2) subalgebra with the components of $\boldsymbol{J}$ restricted to $\mathscr{H}^{(+)}$. We shall denote the latter by M_i, i.e.

$$M_i = P^{(+)} J_i P^{(+)} \tag{3.3}$$

and the remaining so(4) generators by A_i, where $i = 1, 2, 3$. These Hermitian operators satisfy the commutation relations

$$[M_i, M_j] = [A_i, A_j] = \mathrm{i}\varepsilon_{ijk} M_k \qquad [M_i, A_j] = \mathrm{i}\varepsilon_{ijk} A_k. \tag{3.4}$$

In $\mathscr{H}^{(-)}$, the level degeneracies being infinite, there are many possibilities for the symmetry Lie algebra (Moshinsky and Patera 1975). However, to get results closely resembling those obtained for $\mathscr{H}^{(+)}$, $\mathscr{L}^{(-)} = $ so(3, 1) seems the most appropriate one. The so(3, 1) algebra indeed has some ladder irreps containing the infinite sequence of so(3)$\simeq$su(2) irreps $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, and characterised by generalised Young pattern labels $[pq]$, where $p = -1 + \mathrm{i}n$, $q = \frac{1}{2}$, and $n \in \mathbb{N}$, or, in Naimark's notations, by some numbers (k_0, c), where $k_0 = \frac{1}{2}$, and $c = -\mathrm{i}n$ (Naimark 1964, Böhm 1979). Hence, this suggests identifying the eigenstates (2.16) belonging to $\mathscr{H}^{(-)}$ with the basis states of $[-1 + \mathrm{i}n, \frac{1}{2}]$, and the generators of the so(3)$\simeq$su(2) subalgebra of so(3, 1) with the components of $\boldsymbol{J}$ restricted to $\mathscr{H}^{(-)}$. We shall denote the latter by m_i, i.e.

$$m_i = P^{(-)} J_i P^{(-)} \tag{3.5}$$

and the remaining so(3, 1) generators by a_i, where $i = 1, 2, 3$. These Hermitian operators satisfy the commutation relations

$$[m_i, m_j] = -[a_i, a_j] = \mathrm{i}\varepsilon_{ijk} m_k \qquad [m_i, a_j] = \mathrm{i}\varepsilon_{ijk} a_k \tag{3.6}$$

which can be obtained from (3.4) by replacing M_i by m_i, and A_i by $-\mathrm{i}a_i$. Moreover, the operators m_i and a_i commute with the so(4) generators M_i and A_i:

$$[M_i, m_j] = [M_i, a_j] = [A_i, m_j] = [A_i, a_j] = 0. \tag{3.7}$$

Since the total angular momentum components J_i may be written as

$$J_i = M_i + m_i \tag{3.8}$$

the su(2) algebra spanned by these operators is actually a subalgebra of the proposed symmetry Lie algebra

$$\mathrm{so}(4)\oplus\mathrm{so}(3,1) \supset \mathrm{su}(2) \tag{3.9}$$

as it should be.

It remains to construct the operators $\boldsymbol{A}$ and $\boldsymbol{a}$. According to rule (i) above, we should first derive ladder operators connecting the degenerate states associated with definite eigenvalues of H. Such a construction is carried out in the next section.

4. Ladder operators connecting the degenerate eigenstates of the Dirac oscillator

Apart from an irrelevant additive constant, the Dirac oscillator Hamiltonian (2.5) has the same spectrum as the operators $\hat{N}+\hat{J}$ and $\hat{N}-\hat{J}$ in $\mathscr{H}^{(+)}$ and $\mathscr{H}^{(-)}$, respectively. This suggests first considering the auxiliary, spin-independent Hamiltonians

$$H_+ = \hat{N}+\hat{L} \qquad H_- = \hat{N}-\hat{L} \tag{4.1}$$

whose eigenfunctions can be expressed in spherical coordinates as

$$\langle \boldsymbol{r}|Nl\mu\rangle = R_{Nl}(r)\,Y_{l\mu}(\theta,\varphi) \tag{4.2}$$

and correspond to the eigenvalues $N+l$ and $N-l$ respectively. In (4.2), $R_{Nl}(r)$ and $Y_{l\mu}(\theta,\varphi)$ have the same meaning as in (2.8).

Since the auxiliary Hamiltonians $H_\pm$ are but the three-dimensional counterpart of the Hamiltonians $\hat{N}\pm|\hat{M}|$, recently studied by Moshinsky *et al* (1990), and where $\hat{N}$ and $\hat{M}$ are the number operator and the angular momentum of the two-dimensional oscillator, we can take advantage of the experience gained in these simpler problems to construct ladder operators for $H_\pm$. The latter should connect an arbitrary state $|Nl\mu\rangle$ with a degenerate one, either $|N+1, l\mp 1, \mu'\rangle$ or $|N-1, l\pm 1, \mu'\rangle$.

Let us introduce the harmonic oscillator creation and annihilation operators, whose spherical components are defined by

$$\eta_q = 2^{-1/2}(x_q - \mathrm{i}p_q) \qquad \xi_q = (-1)^q\xi^{-q} = (-1)^q(\eta_{-q})^\dagger \qquad q=1,0,-1 \tag{4.3}$$

respectively. By using the values of their reduced matrix elements between two states of type (4.2) (Moshinsky 1969) and su(2) tensor calculus (Rose 1957), it is straightforward to show that the operators

$$F_q = \eta_q(\hat{N}-\hat{L}) - (\boldsymbol{\eta}\cdot\boldsymbol{\eta})\xi_q \qquad G_q = (-1)^q(F_{-q})^\dagger = (\hat{N}-\hat{L})\xi_q - \eta_q(\boldsymbol{\xi}\cdot\boldsymbol{\xi}) \tag{4.4}$$

and

$$f_q = \eta_q(\hat{N}+\hat{L}+1) - (\boldsymbol{\eta}\cdot\boldsymbol{\eta})\xi_q \qquad g_q = (-1)^q(f_{-q})^\dagger = (\hat{N}+\hat{L}+1)\xi_q - \eta_q(\boldsymbol{\xi}\cdot\boldsymbol{\xi}) \tag{4.5}$$

where $q=1,0,-1$, are ladder operators for H_+ and H_- respectively. Their reduced matrix elements between any two eigenstates of these Hamiltonians are indeed given by

$$\begin{aligned}\langle N'l'\|F\|Nl\rangle &= -\delta_{N',N+1}\delta_{l',l-1}(2l+1)\left(\frac{(N-l+2)l}{2l-1}\right)^{1/2}\\ \langle N'l'\|G\|Nl\rangle &= \delta_{N',N-1}\delta_{l',l+1}[(N-l)(l+1)(2l+3)]^{1/2}\end{aligned} \tag{4.6}$$

and

$$\langle N'l'\|f\|Nl\rangle = \delta_{N',N+1}\delta_{l',l+1}(2l+1)\left(\frac{(N+l+3)(l+1)}{2l+3}\right)^{1/2}$$
$$\langle N'l'\|g\|Nl\rangle = -\delta_{N',N-1}\delta_{l',l-1}[(N+l+1)l(2l-1)]^{1/2}. \tag{4.7}$$

Let us now go back to the Dirac oscillator Hamiltonian (2.5). By using definitions (2.13) and (2.16), and su(2) tensor calculus again, we observe that in $\mathscr{H}^{(+)}$ ($\mathscr{H}^{(-)}$) the operators F_q and G_q (f_q and g_q) can only connect an eigenstate $|\nu jm)$ ($|\nu jm]$) of this Hamiltonian with a degenerate eigenstate $|\nu, j-1, m+q)$ or $|\nu, j+1, m+q)$($|n, j+1, m+q]$ or $|n, j-1, m+q]$) since

$$(\nu'j'\|F\|\nu j) = -\delta_{\nu',\nu}\delta_{j',j-1}(j+1)\left(\frac{(2\nu-2j+3)(2j+1)}{j}\right)^{1/2}$$
$$(\nu'j'\|G\|\nu j) = \delta_{\nu',\nu}\delta_{j',j+1}(j+2)\left(\frac{(2\nu-2j+1)(2j+1)}{j+1}\right)^{1/2} \tag{4.8}$$

and

$$[n'j'\|f\|nj] = \delta_{n',n}\delta_{j',j+1}j\left(\frac{(2n+2j+2)(2j+1)}{j+1}\right)^{1/2}$$
$$[n'j'\|g\|nj] = -\delta_{n',n}\delta_{j',j-1}(j-1)\left(\frac{(2n+2j)(2j+1)}{j}\right)^{1/2}. \tag{4.9}$$

We therefore conclude that F_q and G_q (f_q and g_q) are ladder operators for the Dirac oscillator Hamiltonian in $\mathscr{H}^{(+)}$ ($\mathscr{H}^{(-)}$).

In the next section, from these ladder operators, we shall construct the generators A_q and a_q of the Dirac oscillator symmetry Lie algebra. Although, in addition to (4.8) and (4.9), the ladder operators also have non-vanishing matrix elements between some eigenstates of H belonging to $\mathscr{H}^{(+)}$ and some eigenstates belonging to $\mathscr{H}^{(-)}$, this does not matter because the projection operators to be used in the next section will cancel them.

5. Construction of the symmetry Lie algebra generators A_q and a_q

As explained in section 3, the Dirac oscillator eigenstates belonging to $\mathscr{H}^{(+)}$, and corresponding to a given value of ν, should carry an so(4) irrep $[\nu+\frac{1}{2}, \frac{1}{2}]$. The reduced matrix elements of the so(4) generators A_q between two basis states of this irrep, characterised by a given value of the angular momentum, are given by (Biedenharn 1961)

$$(\nu j'\|A\|\nu j) = \delta_{j',j+1}\frac{1}{4}\left(\frac{(2\nu+2j+5)(2\nu-2j+1)(2j+1)}{j+1}\right)^{1/2} + \delta_{j',j}\frac{2\nu+3}{4[j(j+1)]^{1/2}}$$
$$-\delta_{j',j-1}\frac{1}{4}\left(\frac{(2\nu+2j+3)(2\nu-2j+3)(2j+1)}{j}\right)^{1/2}. \tag{5.1}$$

Comparison with equation (4.8) and with the relation (Rose 1957)

$$(\nu j'\|J\|\nu j) = \delta_{j',j}[j(j+1)]^{1/2} \tag{5.2}$$

suggests the following form for A_q:

$$A_q = P^{(+)}\tfrac{1}{4}[(\hat{J}+2)^{-1}(H+2\hat{J}+2)^{1/2}F_q + HJ^{-2}J_q + G_q(H+2\hat{J}+2)^{1/2}(\hat{J}+2)^{-1}]P^{(+)} \tag{5.3}$$

since j, $j(j+1)$, and ε_ν, given in (2.10), are respectively the eigenvalues of $\hat{J}$, $\boldsymbol{J}^2$ and H.

On the other hand, the Dirac oscillator eigenstates belonging to $\mathscr{H}^{(-)}$, and corresponding to a given value of n, should carry an so(3, 1) irrep $[-1+\mathrm{i}n, \frac{1}{2}]$. The reduced matrix elements of the so(3, 1) generators a_q between two basis states of this irrep, characterised by a given value of the angular momentum, can be obtained by analytic continuation of the corresponding result for the so(4) generators A_q and the so(4) irrep $[pq]$ (Biedenharn 1961), i.e. by making the substitutions $A_q \to -\mathrm{i}a_q$, $p \to -1+\mathrm{i}n$, $q \to \frac{1}{2}$. They are given by

$$[nj'\|a\|nj] = \delta_{j',j+1}\frac{1}{2}\left(\frac{[(j+1)^2+n^2](2j+1)}{j+1}\right)^{1/2} - \delta_{j',j}\frac{n}{2[j(j+1)]^{1/2}} - \delta_{j',j-1}\frac{1}{2}\left(\frac{[j^2+n^2](2j+1)}{j}\right)^{1/2}. \tag{5.4}$$

Note that the phase discrepancy with respect to the corresponding Naimark result (Naimark 1964, Böhm 1979) comes from a different phase convention for the irrep basis states. Comparison of (5.4) with equation (4.9) and with a relation similar to (5.2) suggests the following form for a_q:

$$a_q = P^{(-)}\tfrac{1}{4}\{(\hat{J}-1)^{-1}[(H^2+4\hat{J}^2)/(H+2\hat{J})]^{1/2}f_q - HJ^{-2}J_q + g_q[(H^2+4\hat{J}^2)/(H+2\hat{J})]^{1/2}(\hat{J}-1)^{-1}\}P^{(-)} \tag{5.5}$$

where we have now used the fact that j, $j(j+1)$ and ε_n, given in (2.14), are respectively the eigenvalues of $\hat{J}$, $\boldsymbol{J}^2$ and H.

By construction, the set of operators $\boldsymbol{M}$, $\boldsymbol{m}$, $\boldsymbol{A}$ and $\boldsymbol{a}$, defined in (3.3), (3.5), (5.3) and (5.5) respectively, obey the commutation relations (3.4), (3.6) and (3.7). Hence we did succeed in constructing a Lie algebra fulfilling the first three requirements of section 3. It remains to check that the fourth condition is also satisfied.

The so(4) algebra has two independent Casimir operators (Biedenharn 1961)

$$C_1 = \boldsymbol{M}^2 + \boldsymbol{A}^2 \qquad C_2 = \boldsymbol{M}\cdot\boldsymbol{A}. \tag{5.6}$$

Their eigenvalues corresponding to the irrep $[\nu+\frac{1}{2}, \frac{1}{2}]$ are given by

$$\langle C_1\rangle = (\nu+\tfrac{1}{2})(\nu+\tfrac{5}{2}) + \tfrac{1}{4} = \tfrac{1}{4}[\varepsilon_\nu^2 - 3] \qquad \langle C_2\rangle = \tfrac{1}{2}(\nu+\tfrac{3}{2}) = \tfrac{1}{4}\varepsilon_\nu. \tag{5.7}$$

This shows that C_1 and C_2 can be rewritten in terms of the restriction of the Hamiltonian to $\mathscr{H}^{(+)}$,

$$H^{(+)} = P^{(+)}HP^{(+)} \tag{5.8}$$

as

$$C_1 = \tfrac{1}{4}[(H^{(+)})^2 - 3] \qquad C_2 = \tfrac{1}{4}H^{(+)}. \tag{5.9}$$

The so(3, 1) algebra also has two independent Casimir operators (Naimark 1964, Böhm 1979)

$$c_1 = \boldsymbol{m}^2 - \boldsymbol{a}^2 \qquad c_2 = \boldsymbol{m}\cdot\boldsymbol{a}. \tag{5.10}$$

Their eigenvalues corresponding to the irrep $[-1+\mathrm{i}n, \frac{1}{2}]$ can be obtained by analytic continuation of those of C_1 and C_2 for the so(4) irrep $[pq]$ and are given by

$$\langle c_1\rangle = -(n^2+1)+\tfrac{1}{4} = -\tfrac{1}{4}[\varepsilon_n^2+3] \qquad \langle c_2\rangle = -\tfrac{1}{2}n = -\tfrac{1}{4}\varepsilon_n. \tag{5.11}$$

We conclude that c_1 and c_2 can be rewritten in terms of the restriction of the Hamiltonian to $\mathscr{H}^{(-)}$,

$$H^{(-)} = P^{(-)}HP^{(-)} \tag{5.12}$$

as

$$c_1 = -\tfrac{1}{4}[(H^{(-)})^2+3] \qquad c_2 = -\tfrac{1}{4}H^{(-)} \tag{5.13}$$

thus proving that all four requirements of section 3 are satisfied. Note that, apart from the substitution of $H^{(-)}$ for $H^{(+)}$, equation (5.13) only differs from (5.9) by some sign changes.

6. Conclusion

In the present paper, we did show that the symmetry Lie algebra of the Dirac oscillator is the direct sum algebra so(4)$\oplus$so(3, 1). This was suggested by a study of the spectrum degeneracies, but the main problem was actually to explicitly obtain the generators of the algebra. This was done in section 5 by using as an intermediate step the ladder operators of the auxiliary Hamiltonians $\hat{N}\pm\hat{L}$.

The problem of these auxiliary Hamiltonians is quite similar to that of their two-dimensional analogues $\hat{N}\pm|\hat{M}|$, which was considered in Moshinsky *et al* (1990) to show that the determination of the ladder operators for a Hamiltonian with accidental degeneracy is not sufficient to get the generators of its symmetry Lie algebra. The procedure followed in the latter reference to construct the generators served as a framework for the more complex case of the Dirac oscillator Hamiltonian that is analysed in the present paper.

The techniques developed in this work and in the previous one (Moshinsky *et al* 1990) seem to be generalisable to many other problems with accidental degeneracy.

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